Given a and b as real numbers, there is a right cone whose height is a+b units, and it is fixed inside a right cylinder of the same height. In the first situation there is also some water between the cylinder and the cone up to the half height of the cylinder. In the second position, if the cylinder is turned upside down and the height of the water happens to be b units, what is the ratio a / b?

Answer from Mathful

Let's analyze the problem step-by-step.

Problem Restatement:

We have a right circular cylinder and a right cone, both of the same height $h = a + b$.

The cone is placed inside the cylinder.

First situation: The region between the cone and the cylinder is filled with water up to half the height of the cylinder (i.e., height $\frac{a+b}{2}$).

Second situation: The cylinder is turned upside down, and the water level is at height b.

We need to find the ratio $\frac{a}{b}$.

Step 1: Define variables and geometry

Let the height of both the cone and cylinder be $h = a + b$.

Let the radius of the cylinder (and the base of the cone) be R.

The cone is inscribed inside the cylinder, so its base radius is also R.

The volume of the cylinder is:

$$V_{\text{cyl}} = \pi R^2 h = \pi R^2 (a + b)$$

The volume of the cone is:

$$V_{\text{cone}} = \frac{1}{3}\pi R^2 h = \frac{1}{3}\pi R^2 (a + b)$$

Step 2: Volume of water in the first situation

The water fills the space between the cone and the cylinder up to height $\frac{h}{2}$.

The volume of the cylinder up to height $\frac{h}{2}$ is:

$$V_{\text{cyl, half}} = \pi R^2 \cdot \frac{h}{2} = \pi R^2 \frac{a+b}{2}$$

The volume of the cone up to height $\frac{h}{2}$ is the volume of a smaller cone similar to the original cone scaled by height $\frac{h}{2}$.

Step 3: Volume of cone up to height x

The volume of a cone up to height $x \leq h$ is:

$$V_{\text{cone},x} = \frac{1}{3}\pi r_x^2 x$$

where r_x is the radius of the cone at height x.

Since the cone is similar at all heights,

$$r_x = R\frac{x}{h} = R\frac{x}{a+b}$$

So,

$$V_{\text{cone},x} = \frac{1}{3}\pi\left(R\frac{x}{a+b}\right)^2 x = \frac{1}{3}\pi R^2 \frac{x^3}{(a+b)^2}$$

Step 4: Volume of water in the first situation

Water fills the volume between the cylinder and the cone up to height $\frac{h}{2}$, so the volume of water is:

$$V_{\text{water, first}} = V_{\text{cyl, half}} - V_{\text{cone, half}} = \pi R^2 \frac{a+b}{2} - \frac{1}{3}\pi R^2 \frac{\left(\frac{a+b}{2}\right)^3}{(a+b)^2}$$

Simplify the cone volume term:

$$\frac{\left(\frac{a+b}{2}\right)^3}{(a+b)^2} = \frac{(a+b)^3/8}{(a+b)^2} = \frac{a+b}{8}$$

So,

$$V_{\text{water, first}} = \pi R^2 \frac{a+b}{2} - \frac{1}{3}\pi R^2 \frac{a+b}{8} = \pi R^2 (a+b)\left(\frac{1}{2} - \frac{1}{24}\right) = \pi R^2 (a+b)\frac{11}{24}$$

Step 5: Volume of water in the second situation

When the cylinder is turned upside down, the water height is b.

The water occupies the bottom b units of the cylinder.

Since the cone is still inside (now flipped upside down relative to the water), we need to find the volume of water between the cone and the cylinder up to height b.

Step 6: Position of the cone when the cylinder is flipped

Initially, the cone's apex was at the top and base at the bottom.

Now, the cylinder is flipped upside down, so the apex is now at the bottom and base at the top.

The water occupies height b from the bottom.

The cone is inverted relative to the water surface.

Step 7: Volume of the cone inside water in the flipped position

The cone's height is $h = a + b$.

Let x be the height from the apex (bottom) up.

The radius of the cone at height x is:

$$r_x = R\frac{x}{h}$$

The volume of the cone up to height b is:

$$V_{\text{cone},b} = \frac{1}{3}\pi r_b^2 b = \frac{1}{3}\pi \left(R\frac{b}{h}\right)^2 b = \frac{1}{3}\pi R^2 \frac{b^3}{h^2}$$

Step 8: Volume of water in the flipped position

Water volume is:

$$V_{\text{water, second}} = V_{\text{cyl},b} - V_{\text{cone},b} = \pi R^2 b - \frac{1}{3}\pi R^2 \frac{b^3}{(a+b)^2} = \pi R^2 \left(b - \frac{b^3}{3(a+b)^2}\right)$$

Step 9: Equate the volumes of water (water amount doesn't change)

The volume of water is the same in both positions, so:

$$V_{\text{water, first}} = V_{\text{water, second}}$$

$$\pi R^2 (a+b)\frac{11}{24} = \pi R^2 \left(b - \frac{b^3}{3(a+b)^2}\right)$$

Cancel πR^2:

$$(a+b)\frac{11}{24} = b - \frac{b^3}{3(a+b)^2}$$

Multiply both sides by $3(a+b)^2$ to clear denominators:

$$3(a+b)^3\frac{11}{24} = 3b(a+b)^2 - b^3$$

Simplify left side:

$$\frac{33}{24}(a+b)^3 = 3b(a+b)^2 - b^3$$

Or:

$$\frac{11}{8}(a+b)^3 = 3b(a+b)^2 - b^3$$

Bring all terms to one side:

$$\frac{11}{8}(a+b)^3 - 3b(a+b)^2 + b^3 = 0$$

Divide through by $(a+b)^3 \neq 0$:

$$\frac{11}{8} - 3\frac{b}{a+b} + \left(\frac{b}{a+b}\right)^3 = 0$$

Let:

$$x = \frac{b}{a+b}$$

Rewrite:

$$\frac{11}{8} - 3x + x^3 = 0$$

Or:

$$x^3 - 3x + \frac{11}{8} = 0$$

Step 10: Solve cubic equation for x

Cubic:

$$x^3 - 3x + \frac{11}{8} = 0$$

Try rational roots (possible roots are factors of $\frac{11}{8}$):

Try $x = 1$:

$$1 - 3 + \frac{11}{8} = 1 - 3 + 1.375 = -0.625 \neq 0$$

Try $x = \frac{1}{2} = 0.5$:

$$(0.5)^3 - 3(0.5) + \frac{11}{8} = 0.125 - 1.5 + 1.375 = 0$$

So $x = \frac{1}{2}$ is a root.

Step 11: Find a/b

Recall:

$$x = \frac{b}{a+b} = \frac{1}{2}$$

So:

Thus,

$$\boxed{\frac{a}{b} = 1}$$

Final answer:

$$\boxed{\frac{a}{b} = 1}$$

Interpretation: The heights a and b are equal for the given water level conditions.

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